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Periodic solutions of the pendulum

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Abstract. We develop an analytical procedure to determine orbits that a harmonically driven, damped pendulum describes in the phase plane. The theory predicts the existence of more than one solution for the same system, depending on initial conditions. Also, it predicts a stable solution around the top position of the pendulum. Trajectories obtained by numerically integrating the pendulum equation in a phase-locked condition agree with our diagrams. Some periodic solutions were found that are not orbitally stable.

1. Introduction

The differential equation that describes the dynamical behaviour of a harmonically driven, linearly damped pendulum in a gravitational field is

$$\ddot{\theta} + \sigma\dot{\theta} + (\omega_0)^2 \sin\theta = A \sin\omega_{\rm d}\tau \tag{1.1}$$

where θ is the angular displacement from the vertical resting axis at a time τ; ω_0 is the undamped natural frequency for small-amplitude oscillations; $\omega_{\rm d}$ is the frequency of the driving and σ and A are the damping constant and the amplitude of the external driving, respectively, in appropriate units. This equation appears as valid for the description of other physical systems, such as Josephson junctions [1–3], phase-locked loops (PLLs) [4, 5] and charge-density wave (CDW) conductors [6, 7].

For small oscillation amplitudes, the simple pendulum can be treated as a harmonic oscillator obeying a linear differential equation and its dynamics can be theoretically predicted. However, for large amplitudes several dynamical scenarios can be found, such as development of chaotic solutions via period doubling cascades, intermittency, hysteresis, crisis etc [8–10]. These nonlinear behaviours are usually determined through computer simulations and experimental measurements and they are usually characterized by numerically calculating diagram bifurcations, Poincaré maps, Lyapunov exponents, winding numbers, power spectra and Renyi's generalized dimensions [11–13]. Nonlinear behaviours are almost intractable, analytically.

An interesting phenomenon which can appear in the dynamics of a harmonically driven pendulum is phase-locking, which happens when its oscillation frequency ν becomes a rational multiple of the frequency $\omega_{\rm d}$ of the driving torque [14]. The motion projection onto a phase plane described by a phase-locked pendulum is a closed orbit, because such a motion is periodic. Phase-locking has been experimentally detected and numerically computed in several works involving forced pendulums [8, 9, 15], driven Josephson junctions [16], driven CDW conductors [17], master–slave PLL networks [18] etc.

The usual method for determining approximated phase-locked solutions consists of expanding $\theta(\tau)$ in Fourier series [19]. However, this idea has not been fully developed.

This is what we are proposing to do here. The coupling effect among different harmonics which occurs through a nonlinear term in the equation is made clear in this theory. The pendulum equation is numerically integrated and the phase diagrams are compared to those obtained by using the theory. Some periodic solutions were found that are not necessarily orbitally stable.

The pendulum equation is rewritten, in dimensionless form, as

$$f^2\ddot{\theta} + \epsilon f\dot{\theta} + \sin\theta = a\sin t \tag{1.2}$$

where $f \equiv \omega_{\rm d}/\omega_0$, $\epsilon \equiv \sigma/\omega_0$, $a \equiv A/(w_0)^2$ and $t \equiv \omega_{\rm d}\tau$. The dot now means derivation with respect to the dimensionless variable t.

The main ideas of the theory are exposed in section 2.

In section 3, an iterative method is established in order to numerically solve the equations obtained in section 2, when the oscillation frequency is the same as the driving frequency. In section 4, we find a sufficient condition for stability. Numerical results are presented as figures. In sections 5 and 6 we discuss our findings.

2. Periodic solutions

Periodic solutions of (1.2) are represented by a Fourier series in the form

$$\theta = \Theta_0 + \sum_{m=1}^{\infty} C_m \sin\left(\frac{m}{p}t - \delta_m\right) \tag{2.1}$$

where p is an integer number with the meaning of the oscillation period relative to the driving period. Thus, for an oscillation period equal to three times the driving period we have $p = 3$.

Θ_0 is the average position and C_m and δ_m are constants representing the amplitude and phase of the harmonic oscillation with frequency m/p.

If this expression is substituted into (1.2), the linear terms are Fourier series obtained by straightforward derivation; the nonlinear term $\sin\theta$ will also be periodic and it can be written as another Fourier series by using the exponential form of $\sin\theta$ and the identity for cylindrical Bessel functions:

$$e^{iq\sin x} = \sum_{\ell=-\infty}^{+\infty} J_\ell(q)e^{i\ell x}. \tag{2.2}$$

$J_\ell(q)$ is a cylindrical Bessel function of order ℓ.

We write

$$\begin{aligned} e^{i\theta} &= e^{i\Theta_0}\prod_{m=1}^{\infty} e^{iC_m\sin(\frac{m}{p}t-\delta_m)} \\ &= e^{i\Theta_0}\sum_{\ell_1=-\infty}^{+\infty} J_{\ell_1}(C_1)e^{i\ell_1(t/p-\delta_1)}\sum_{\ell_2=-\infty}^{+\infty} J_{\ell_2}(C_2)e^{i\ell_2(2t/p-\delta_2)} \\ &\quad\times\sum_{\ell_3-\infty}^{+\infty} J_{\ell_3(C_3)}e^{i\ell_3(3t/p-\delta_3)}\ldots \\ &= e^{i\Theta_0}\sum_{s=-\infty}^{+\infty} e^{ist/p}\left[\sum\nolimits^{s} J_{\ell_1}(C_1)J_{\ell_2}(C_2)J_{\ell_3}(C_3)\ldots e^{-i(\ell_1\delta_1+\ell_2\delta_2+\ell_3\delta_3+\cdots)}\right] \end{aligned} \tag{2.3}$$

where $\sum^s$ is the sum over all possible values of $\{\ell_1, \ell_2, \ldots\}$ that satisfy

$$\ell_1 + 2\ell_2 + 3\ell_3 + \cdots = s.$$

We use the notations

$$\ell_t \equiv \sum_{m=1}^{\infty} \ell_m \qquad \text{and} \qquad J^{\ell_t} \equiv J_{\ell_1}(C_1)J_{\ell_2}(C_2)\ldots$$

to make the expressions brief, whenever there is no problem of confusion.

Also, some indices are omitted for the same reason. Thus

$$\sum_{m=1}^{\infty} \ell_m \equiv \sum \ell_m \qquad \sum_{m=1}^{\infty} m\ell_m \equiv \sum m\ell_m \qquad \text{and} \qquad \sum_{m=1}^{\infty} \ell_m\delta_m \equiv \sum \ell_m\delta_m.$$

By assembling all the contributions of Fourier series in (1.2) and matching the two sides of the equation we obtain a set of equations, containing C_m and δ_m, one for each harmonic ($s = 0$, $s = p$ and other values of s).

The algebra is reported in the appendix.

Non-oscillatory term (s = 0)

$$\sin\Theta_0 \sum_{\ell_t=\text{even}}^{0} J_{\ell_1}(C_1)J_{\ell_2}(C_2)\ldots\cos\left(\sum_{m=1}^{\infty}\ell_m\delta_m\right) - \cos\Theta_0 \sum_{\ell_t=\text{odd}}^{0} J_{\ell_1}(C_1)J_{\ell_2}(C_2)\ldots\sin\left(\sum_{m=1}^{\infty}\ell_m\delta_m\right) = 0 \tag{2.4}$$

$\sum_{\ell_t=\text{even}}{}^{s}$ is the sum over all possible values of $\{\ell_1, \ell_2, \ldots\}$ that satisfy $\ell_1 + 2\ell_2 + 3\ell_3 + \cdots = s$ and $\ell_t = \text{even}$.

Analogous notation is used for $\ell_t = \text{odd number}$.

Harmonic p

$$(-f^2 + \mathrm{i}\epsilon f)C_p\mathrm{e}^{-\mathrm{i}\delta_p} + 2\mathrm{i}\sin\Theta_0 \sum_{\ell_t=\text{even}}^{p} J^{\ell_t}\mathrm{e}^{-\mathrm{i}\sum\ell_m\delta_m} + 2\cos\Theta_0 \sum_{\ell_t=\text{odd}}^{p} J^{\ell_t}\mathrm{e}^{-\mathrm{i}\sum\ell_m\delta_m} = a. \tag{2.5}$$

There is a term in $\sum^p$ that is zero if $C_p = 0$ and is $\neq 0$ if $C_p \neq 0$, even if all the other coefficients C_m ($m \neq p$) are zero. This is

$$2\cos\Theta_0 J_0(C_1)J_0(C_2)\ldots J_1(C_p)\ldots\mathrm{e}^{-\mathrm{i}\delta_p} \tag{2.6}$$

($\ell_p = 1$; $\ell_m = 0$ for $m \neq p$; $\sum m\ell_m = p$). It is convenient to reorganize (2.5) by isolating this term.

We call $j_p(C)$ the product of all zero-order Bessel functions but $J_0(C_p)$ and $D_p(C,\delta)$ the nonlinear part, excluding (2.6):

$$j_p(C) \equiv J_0(C_1)J_0(C_2)\ldots J_0(C_p)\ldots J_0(C_m)\ldots/J_0(C_p) \tag{2.7}$$

$$-D_p(C,\delta) \equiv 2\mathrm{i}\sin\Theta_0 \sum_{\ell_t=\text{even}}^{p} J^{\ell_t}\mathrm{e}^{-\mathrm{i}\sum\ell_m\delta_m} + 2\cos\Theta_0 \sum_{\ell_t=\text{odd}}^{p} J^{\ell_t}\mathrm{e}^{-\mathrm{i}\sum\ell_m\delta_m} - 2\cos\Theta_0 j_p(C)J_1(C_p)\mathrm{e}^{-\mathrm{i}\delta_p} \tag{2.8}$$

$$C \equiv \{C_1, C_2, C_3, \ldots\} \qquad \text{and} \qquad \delta \equiv \{\delta_1, \delta_2, \delta_3, \ldots\}.$$

Equation (2.5) becomes

$$(-f^2 + \mathrm{i}\epsilon f)C_p + 2\cos\Theta_0 j_p(C)J_1(C_p) = (a + D_p(C,\delta))\mathrm{e}^{\mathrm{i}\delta_p}. \tag{2.9}$$

If the right-hand side of this equation is different from zero, C_p must be different from zero.

Harmonic $s \neq p$

$$\left(-\left(\frac{fs}{p}\right)^2 + \mathrm{i}\epsilon\frac{fs}{p}\right) C_s \mathrm{e}^{-\mathrm{i}\delta_s} + 2\mathrm{i}\sin\Theta_0 \sum_{\ell_t=\text{even}}^{s} J^{\ell_t}\mathrm{e}^{-\mathrm{i}\sum \ell_m\delta_m} + 2\cos\Theta_0 \sum_{\ell_t=\text{odd}}^{s} J^{\ell_t}\mathrm{e}^{-\mathrm{i}\sum \ell_m\delta_m} = 0. \tag{2.10}$$

This equation is also reorganized in a similar way. Using the same notations

$$\left(-\left(\frac{fs}{p}\right)^2 + \mathrm{i}\epsilon\frac{fs}{p}\right) C_s + 2\cos\Theta_0 j_s(C) J_1(C_s) = D_s(C,\delta)\mathrm{e}^{\mathrm{i}\delta_s}. \tag{2.11}$$

If $C_j = 0$, then $J_{\ell_j}(C_j) = 0$ unless $\ell_j = 0$. So, if we restrict the number of coefficients ($C_j = 0$ for $j > M$), only terms with $\ell_j = 0$, $j > M$ will contribute to $D_s(C,\delta)$. Thus, $D_s(C,\delta)$ will no longer contain an infinite product of Bessel functions.

2.1. Small oscillations

For small oscillations ($C_1, C_2 \ldots \ll 1$) around equilibrium ($\Theta_0 \simeq 0$), $j_s(C) \simeq 1$ and $J_1(C_s) \simeq C_s/2$ and (2.9) and (2.11) are reduced to

$$(-f^2 + \mathrm{i}\epsilon f + 1)C_p \mathrm{e}^{-\mathrm{i}\delta_p} \simeq a$$

and

$$\left(-\left(\frac{fs}{p}\right)^2 + \mathrm{i}\epsilon\frac{fs}{p} + 1\right) C_s \mathrm{e}^{-\mathrm{i}\delta_s} \simeq 0 \qquad \text{for} \quad s \neq p.$$

This is the equation of a damped linear pendulum, which allows for the stationary solution:

$$C_s = 0 \qquad \text{for} \quad s \neq p$$
$$C_p \simeq \frac{a}{\sqrt{(1-f^2)^2 + \epsilon^2 f^2}}$$
$$\delta_p = \tan^{-1}\frac{\epsilon f}{1-f^2}.$$

2.2. Harmonics

For large-amplitude oscillations, the terms D_s become important. These terms represent the coupling among different harmonic oscillations.

For illustration, table 1 is given to show all the possible combinations of $\{\ell_1, \ell_2, \ldots, \ell_6\}$, with ℓ_t = odd number and $s \equiv \sum m\ell_m = 2$ and $s = 3$, assuming

$$|\ell_i| \leqslant 4 \qquad \text{for} \quad i = 1, 2, \ldots, 6 \qquad \text{and} \qquad \sum_{i=1}^{6} |\ell_i| \leqslant 4.$$

These assumptions are reasonable if only the first six coefficients are relevant and the Bessel functions of order higher than four are negligible.

For $s = 2$, for example, the first row means $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\} = \{-1, -1, 0, 0, 0, 1\}$. Its contribution to $D_2(C,\delta)$ is

$$-2\cos\Theta_0 J_{-1}(C_1) J_{-1}(C_2) J_0(C_3) J_0(C_4) J_0(C_5) J_1(C_6) \mathrm{e}^{\mathrm{i}(\delta_1+\delta_2-\delta_6)}.$$

This term yields second harmonics.

Table 1. $s = 2$; $\ell_t = \text{odd}$; $|\ell_i| \leqslant 4$.

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
0	−2	0	0	0	1
−1	−1	0	0	1	0
1	−1	1	0	0	0
−1	0	−1	0	0	1
1	0	−1	1	0	0
1	0	0	−1	1	0
1	0	0	0	−1	1
−2	0	0	1	0	0
0	0	0	2	0	−1
0	0	1	0	1	−1
0	0	1	1	−1	0
0	0	2	−1	0	0
0	1	0	0	0	0

Table 2. $s = 3$; $\ell_t = \text{odd}$; $|\ell_i| \leqslant 4$.

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
−1	−1	0	0	0	1
1	0	−1	0	1	0
1	0	0	−1	0	1
3	0	0	0	0	0
−2	0	0	0	1	0
0	0	0	1	1	−1
0	0	0	2	−1	0
0	0	1	0	0	0
0	1	−1	1	0	0
0	1	0	−1	1	0
0	1	0	0	−1	1
−1	2	0	0	0	0

2.3. Energy balance

Another equation, which is not independent from (1.2) (and, therefore, from (2.4), (2.9) and (2.11)), that may be useful is the energy balance equation. It can be derived by multiplying (1.2) by $\dot{\theta}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f^2\frac{\dot{\theta}^2}{2} - \cos\theta\right) + \epsilon f\dot{\theta}^2 = \dot{\theta}a\sin t.$$

We seek for periodic motions with period $p \times 2\pi$. Integration in one period of the first term (total energy) is 0 and the result is that, in one period, the energy dissipation is equal to energy input:

$$\int_0^{2p\pi} \epsilon f\dot{\theta}^2\,\mathrm{d}t = \int_0^{2p\pi} \dot{\theta}a\sin t\,\mathrm{d}t.$$

Using (2.1) we obtain

$$aC_p\sin\delta_p = \epsilon f\sum_{s=1}^{\infty}\left(\frac{sC_s}{p}\right)^2. \tag{2.12}$$

3. Iteration method (p = 1 solutions)

We consider stationary solutions with fundamental frequency of oscillation equal to the frequency of the driving force ($p = 1$).

If C_i are small, $D_1(C, \delta)$ in (2.9) is small. As $a \neq 0$, C_1 must be $\neq 0$; that is to say, the external force directly drives the oscillation with unit frequency. $D_1(C, \delta)$ represents the feedback due to other harmonics.

According to (2.11), if $D_s(C, \delta) \neq 0$, $s \neq 1$, this term yields oscillation with frequency s. Therefore $D_s(C, \delta)$ is the driving term of each one of other harmonics. This coupling effect among different harmonics arises due to the nonlinear term in the original equation.

In this paper we assume the following.

- (i) $C_i \geqslant 0$. This is not restrictive as the sign can be absorbed by the argument δ_i.
- (ii) $C_1 \gg C_i$ for $i \neq 1$. This is expected to be true as $i \neq 1$ oscillation is not directly driven by the external force. This assumption may fail if some kind of resonance occurs at higher frequency. This did not happen in our calculations.
- (iii) $C_i = 0$ for $i > 6$. The number 6 was chosen arbitrarily. We did not expect very high-order harmonics; we thought it would be sensible to take at least three odd-order harmonics. *A posteriori*, we found this was a good guess.
- (iv) $|\ell_1| \leqslant 7$ and $|\ell_i| \leqslant 3$ for $i \neq 1$. This means that we neglect Bessel functions $J_{\ell_1}(C_1)$ and $J_{\ell_i}(C_i)$ for $|\ell_1| > 7$ and $|\ell_i| > 3$, $i \neq 1$. In our calculations higher values of ℓ_i did not improve the results, except for one case, shown in figure 13 below.

We establish an iterative procedure: we start from approximate values of the coefficients

$$C^0 \equiv \{C_1^0, C_2^0, \ldots, C_6^0\} \qquad \text{and} \qquad \delta^0 \equiv \{\delta_1^0, \delta_2^0, \ldots, \delta_6^0\}$$

to find more accurate values

$$C \equiv \{C_1, C_2, \ldots, C_6\} \qquad \text{and} \qquad \delta \equiv \{\delta_1, \delta_2, \ldots, \delta_6\}.$$

3.1. First-order approximation

3.1.1. Average position Θ_0. We write (2.4) with

$$C_m^0 \equiv 0 \qquad m \neq 1$$

in order to find Θ_0.

As $J_\ell(0) = 0$ for $\ell \neq 0$, only one term may be different from zero: $\ell \equiv \{\ell_1, 0, 0, 0, 0, 0\}$. As $\sum m\ell_m \equiv s = 0$, also $\ell_1 = 0$. Equation (2.4) reduces to

$$\sin\Theta_0 J_0(C_1^0) = 0.$$

Thus, either $\Theta_0 = 0$ or $\Theta_0 = \pi$.

Phase paths of periodic motions surround either the lowest position of the pendulum (stable equilibrium point) or the top position (unstable equilibrium point).

3.1.2. Equation for $s = 1$. Equation (2.9) is written

$$(-f^2 + \mathrm{i}\epsilon f)C_1 + 2\cos\Theta_0 j_1(C^0)J_1(C_1) = (a + D_1(C', \delta^0))\mathrm{e}^{\mathrm{i}\delta_1} \qquad (3.1)$$

where

$$C' \equiv \{C_1, C_2^0, C_3^0, C_4^0, C_5^0, C_6^0\}.$$

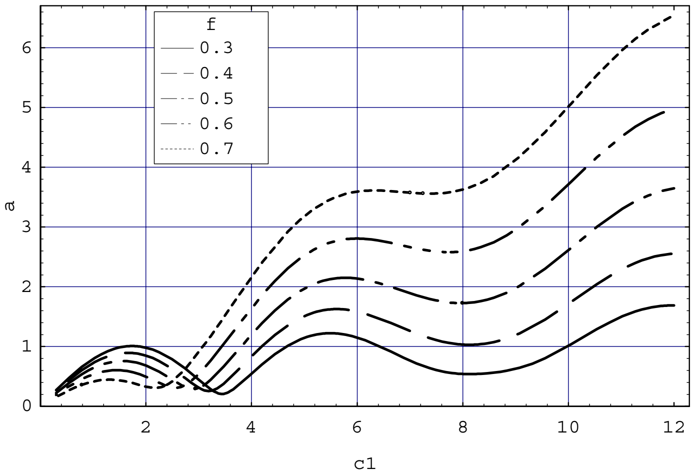

Figure 1. First-order approximation. $\epsilon = 0.2$; $\Theta_0 = 0$.

C_1 is obtained from

$$|(-f^2 + \mathrm{i}\epsilon f)C_1 + 2\cos\Theta_0 j_1(C^0)J_1(C_1)| = |a + D_1(C', \delta^0)| \tag{3.2}$$

and δ_1 from

$$\delta_1 = \mathrm{Arg}\left\{\frac{(-f^2 + \mathrm{i}\epsilon f)C_1 + 2\cos\Theta_0 j_1(C^0)J_1(C_1)}{a + D_1(C', \delta^0)}\right\}. \tag{3.3}$$

Alternatively, δ_1 can be obtained using the energy balance equation (2.12):

$$aC_1 \sin\delta_1 = \epsilon f \sum_{s=1}^{\infty}(sC_s)^2.$$

In our calculations, the results were found to be consistent.

Again, as $C_m^0 = 0$, for $m \neq 1$, $J_{\ell_m}(C_m^0) = 0$ unless $\ell_m = 0$; $s = \sum m\ell_m = \ell_1 = 1$ and $\ell_t = 1$. Therefore

$$D_1(C', \delta^0) = 0.$$

Figures 1 and 2 show a as a function of C_1 for $\epsilon = 0.2$ and $\Theta_0 = 0$ and $\Theta_0 = \pi$ respectively.

In principle, the same force a may yield different oscillations C_1.

3.1.3. Equation for $s = 2$. We write (2.11) as

$$(-(2f)^2 + \mathrm{i}2\epsilon f)C_2 + 2\cos\Theta_0 j_2(C')J_1(C_2) = D_2(C'', \delta')\mathrm{e}^{\mathrm{i}\delta_2} \tag{3.4}$$

where $C'' \equiv \{C_1, C_2, C_3^0, C_4^0, C_5^0, C_6^0\}$ and $\delta' \equiv \{\delta_1, \delta_2^0, \delta_3^0, \delta_4^0, \delta_5^0, \delta_6^0\}$.

$C_2 \neq 0$ if $D_2(C'', \delta') \neq 0$. $D_2(C'', \delta')$ is given by

$$D_2(C'', \delta') = -2\cos\Theta_0 \sum_{\ell_t = \mathrm{odd}} J_{\ell_1}(C_1)J_{\ell_2}(C_2)J_{\ell_3}(C_3^0)\ldots\mathrm{e}^{-\mathrm{i}\sum \ell_m\delta_m}.$$

Only terms with $\ell_3 = \cdots = \ell_6 = 0$ survive. We see in the table 1 that there are none, in this approximation. Thus

$$D_2(C'', \delta') = 0 \qquad \text{and} \qquad \text{therefore} \quad C_2 = 0. \tag{3.5}$$

If higher values of ℓ_i are taken, C_2 may become $\neq 0$, but, still very small as $|J_{\ell_1}(C_1)| \ll 1$, for $|\ell_1| \gg 1$.

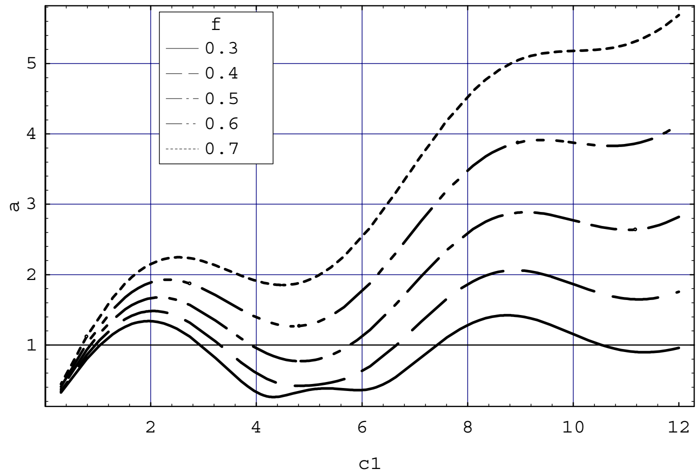

Figure 2. First-order approximation. $\epsilon = 0.2$; $\Theta_0 = \pi$.

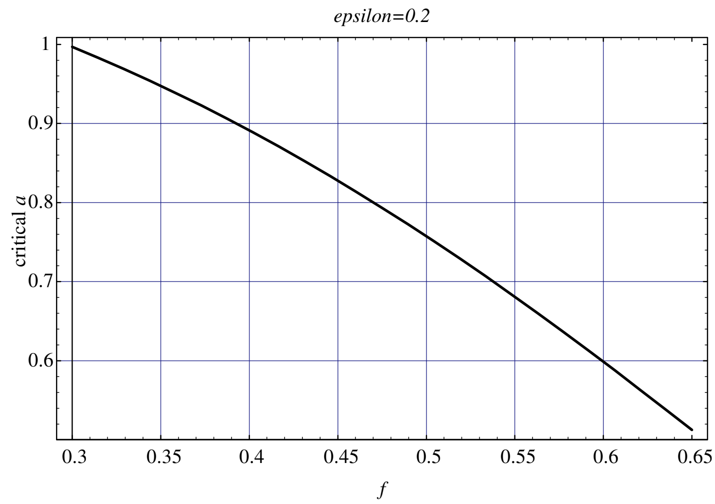

Figure 3. First-order approximation. Stable $p = 1$ solution below the curve.

3.1.4. Equation for $s \geqslant 3$. We write (2.11) for $s = 3$ using

$$C''' \equiv \{C_1, C_2, C_3, C_4^0, C_5^0, C_6^0\} \qquad \text{and} \qquad \delta'' \equiv \{\delta_1, \delta_2, \delta_3^0, \delta_4^0, \delta_5^0, \delta_6^0\}.$$

In table 1, immediately we see the set of values $\ell \equiv \{3, 0, 0, 0, 0, 0\}$ that corresponds to a non-zero driving term:

$$D_3(C''', \delta'') \simeq -2\cos\Theta_0 J_3(C_1) J_0(C_3) \mathrm{e}^{-3\mathrm{i}\delta_1}.$$

Therefore, $C_1 \neq 0$ yields oscillation with frequency 3. C_3 is the solution of

$$\left(-\left(\frac{3f}{p}\right)^2 + \mathrm{i}\epsilon\frac{3f}{p}\right) C_3 + 2\cos\Theta_0 j_3(C'') J_1(C_3) = D_3(C''', \delta'')\mathrm{e}^{\mathrm{i}\delta_3}. \qquad (3.6)$$

$$\delta^3 = \mathrm{Arg}\left\{\frac{(-(\frac{3f}{p})^2 + \mathrm{i}\epsilon\frac{3f}{p})C_3 + 2\cos\Theta_0 j_3(C'') J_1(C_3)}{D_3(C''', \delta'')}\right\}. \qquad (3.7)$$

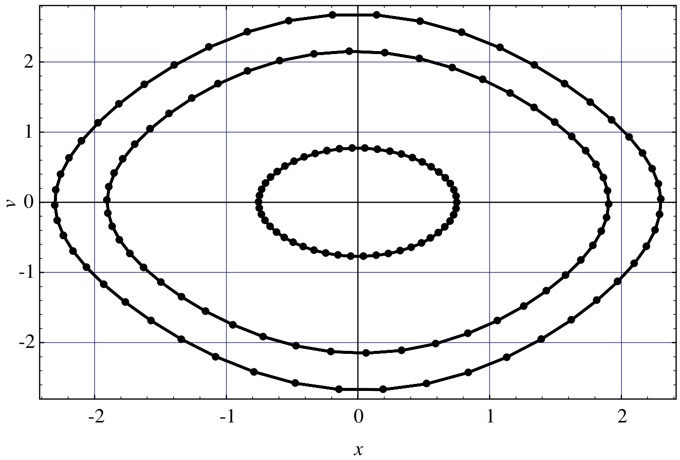

Figure 4. $\epsilon = 0.2$; $a = 0.35$; $f = 0.7$. $C_{\rm odd} = \{0.757, 0.004\,89, 0\}$; $\delta_{\rm odd} = \{0.308, 0.756, 0\}$; $\{x_0, v_0\} = \{-0.233, 0.732\}$. $C_{\rm odd} = \{1.96, 0.0588, 0.002\,71\}$; $\delta_{\rm odd} = \{0.912, 2.64, 4.44\}$; $\{x_0, v_0\} = \{-1.58, 1.04\}$. $C_{\rm odd} = \{2.38, 0.0886, 0.005\,57\}$; $\delta_{\rm odd} = \{1.83, 5.4, 2.78\}$; $\{x_0, v_0\} = \{-2.24, -0.475\}$.

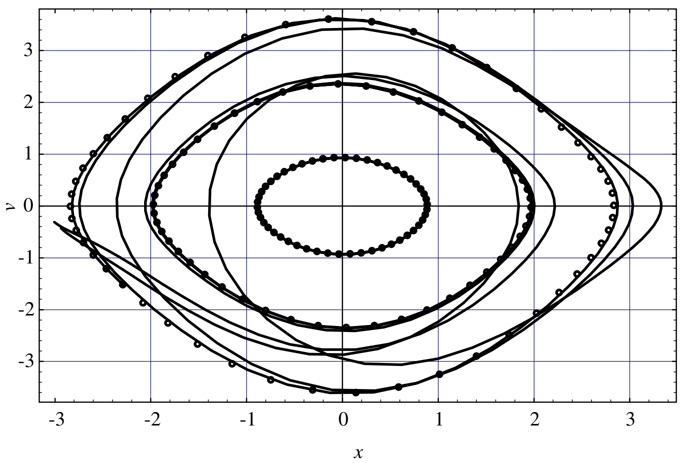

Figure 5. $\epsilon = 0.2$; $a = 0.5$; $f = 0.6$. $C_{\rm odd} = \{0.898, 0.0117, 0, 0\}$; $\delta_{\rm odd} = \{0.218, 0.506, 0\}$; $\{x_0, v_0\} = \{-0.2, 0.907\}$. $C_{\rm odd} = \{2.06, 0.0901, 0.0054\}$; $\delta_{\rm odd} = \{0.526, 1.46, 2.48\}$; $\{x_0, v_0\} = \{-1.12, 1.79\}$; unstable. $C_{\rm odd} = \{2.99, 0.175, 0.018\}$; $\delta_{\rm odd} = \{2.31, 0.534, 5.14\}$; $\{x_0, v_0\} = \{-2.29, -1.52\}$; unstable.

A similar analysis leads to the conclusion that $C_4 = C_6 = 0$ and that C_3 and C_5 can be $\neq 0$.

3.2. Higher-order approximation

We assume for C^0 and δ_0 the newly calculated values. All even-order terms are null.

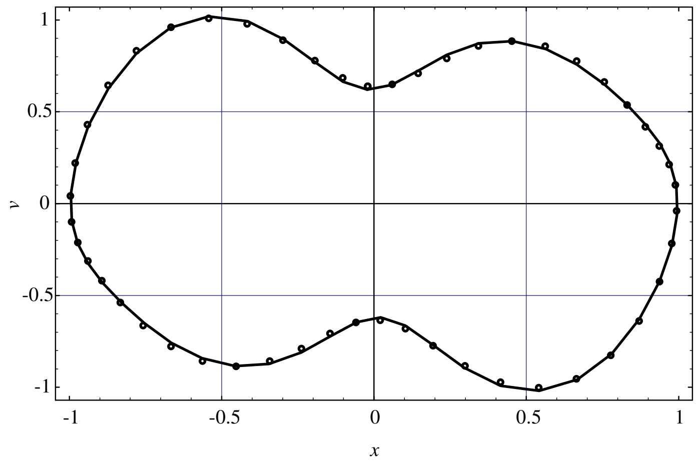

Figure 6. $\epsilon = 0.2$; $a = 0.8$; $f = 0.2$. $C_{\mathrm{odd}} = \{0.947, 0.069, 0.025\}$; $\delta_{\mathrm{odd}} = \{0.0505, -2.68, 2.99\}$; $\{x_0, v_0\} = \{-0.0196, 0.637\}$.

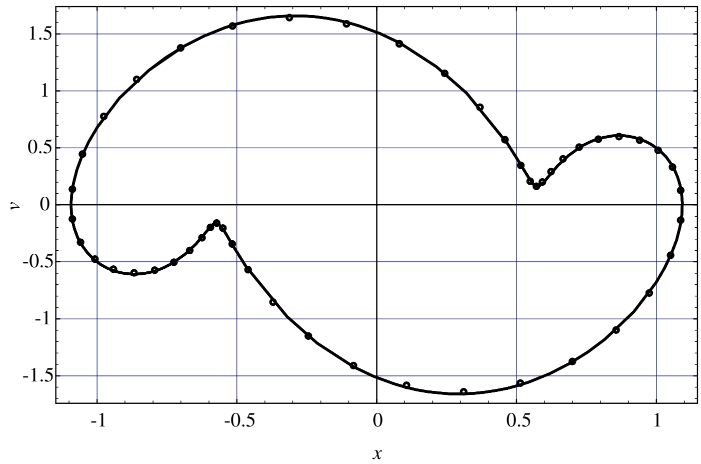

Figure 7. $\epsilon = 0.2$; $a = 0.8$; $f = 0.3$. $C_{\mathrm{odd}} = \{1.01, 0.213, 0.016\}$; $\delta_{\mathrm{odd}} = \{0.107, -0.996, 5.51\}$; $\{x_0, v_0\} = \{0.904, 0.589\}$.

3.2.1. Average position Θ_0. Only terms $\ell_2 = \ell_4 = \ell_6 = 0$ are relevant in the $s = 0$ equation (2.4). We may write

$$s \equiv \sum m\ell_m = \ell_t + (\ell_2 + 3\ell_4 + 5\ell_6 + \cdots) + (2\ell_3 + 4\ell_5 + 6\ell_7 + \cdots).$$

So

$$\text{parity of } s = \text{parity of } \ell_t + \text{parity of } \sum \ell_{2j}.$$

ℓ_t must be an even number and from (2.4) we draw the conclusion that, if we do not have even-order coefficients ($C_{2j} = 0$),

$$\Theta_0 = 0 \qquad \text{or} \qquad \Theta_0 = \pi. \tag{3.8}$$

3.2.2. Equation for s = 1. C_1 is modulated by a feedback effect related to $D_1(C', \delta^0)$.

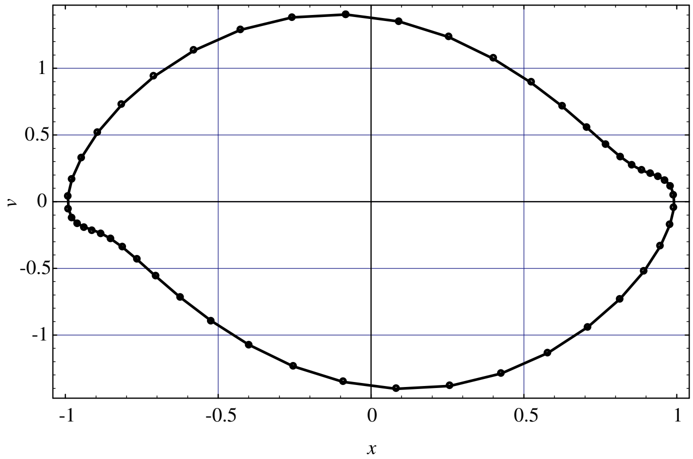

Figure 8. $\epsilon = 0.2$; $a = 0.8$; $f = 0.35$. $C_{\text{odd}} = \{1.06, 0.11, 0.0057\}$; $\delta_{\text{odd}} = \{0.102, -0.225, -0.094\}$; $\{x_0, v_0\} = \{-0.0825, 1.4\}$.

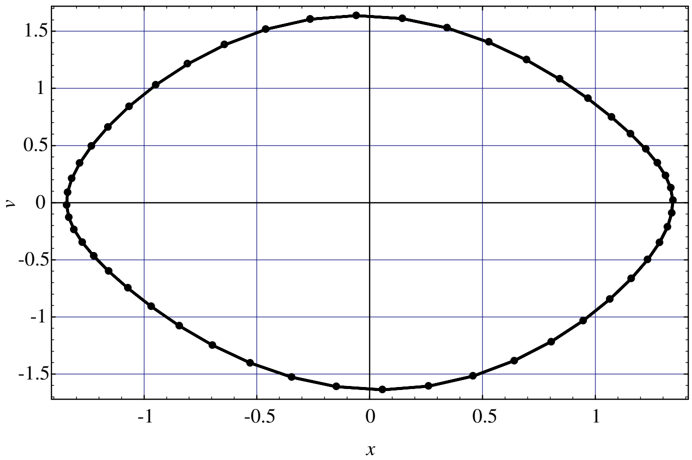

Figure 9. $\epsilon = 0.2$; $a = 0.8$; $f = 0.47$. $C_{\text{odd}} = \{1.41, 0.071, 0.0034\}$; $\delta_{\text{odd}} = \{0.17, 0.314, 0.599\}$; $\{x_0, v_0\} = \{-0.263, 1.61\}$.

3.2.3. Equation for $s = 2$. Relevant terms have $\ell_4 = \ell_6 = 0$ and they are found, in table 1, to be

$$\{-1, -1, 0, 0, 1, 0\} \qquad \text{and} \qquad \{1, -1, 1, 0, 0, 0\}.$$

$D_2(C'', \delta')$ can be $\neq 0$ and therefore C_2 can be $\neq 0$. All harmonics can be excited in different degrees.

3.2.4. Equation for $s \geqslant 3$. The same procedure is repeated until the solution converges.

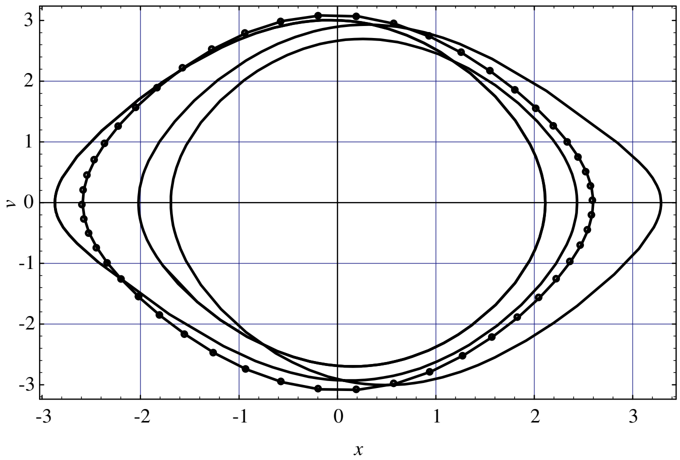

Figure 10. $\epsilon = 0.2$; $a = 0.52$; $f = 0.689$. $C_{\text{odd}} = \{2.7, 0.115, 0.0089\}$; $\delta_{\text{odd}} = \{2.33, 0.609, 5.26\}$; $\{x_0, v_0\} = \{-2.02, -1.55\}$. $\{x_0, v_0\} = \{-2.32, -1.78\}$ results in $p = 3$.

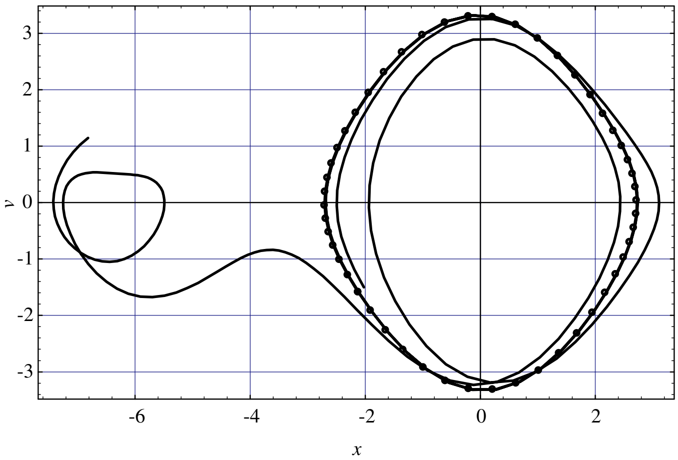

Figure 11. $\epsilon = 0.2$; $a = 0.52$; $f = 0.65$. $C_{\text{odd}} = \{2.84, 0.139, 0.0123\}$; $\delta_{\text{odd}} = \{2.33, 0.609, 5.26\}$; $\{x_0, v_0\} = \{-2.13, -1.58\}$. $\{x_0, v_0\} = \{-2.02, -1.5\}$ results in chaotic motion.

4. Stability of a solution

Let $\bar{\theta}(t)$ be a stationary solution of (1.2). A deviation $\delta\theta$ from $\bar{\theta}(t)$ satisfies the equation

$$f^2\delta\ddot{\theta} + \epsilon f\delta\dot{\theta} + \sin(\bar{\theta} + \delta\theta) - \sin\bar{\theta} = 0. \quad (4.1)$$

For small $\delta\theta$ this equation is linearized to

$$f^2\delta\ddot{\theta} + \epsilon f\delta\dot{\theta} + \cos\bar{\theta}\delta\theta \simeq 0. \quad (4.2)$$

$|\delta\theta|$ is a decreasing function of t if $\cos\bar{\theta} > 0$. Therefore if $\cos\bar{\theta} > 0$ the solution is stable.

For $p = 1$ oscillations, assuming that $C_1 \gg C_m$, $m \neq 1$, we may say that for $\Theta_0 = 0$ and $C_1 \leqslant \pi/2$ the solution is stable.

The behaviour of the system is determined by the dimensionless damping factor ϵ, the driving frequency f and the driving amplitude a.

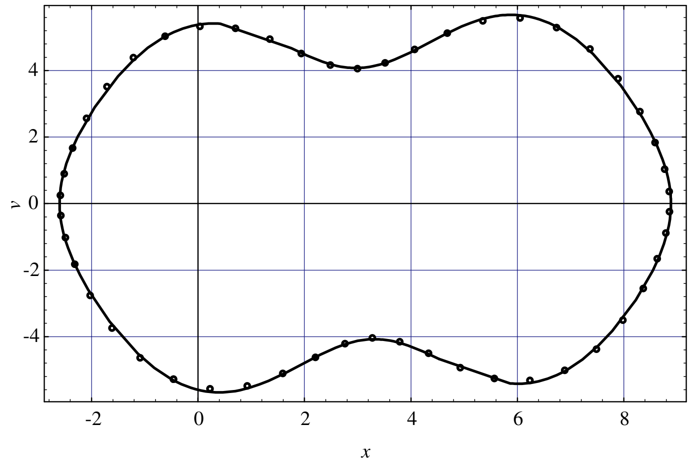

Figure 12. $\Theta_0 = \pi$; $\epsilon = 0.2$; $a = 0.78$; $f = 0.45$. $C_{\text{odd}} = \{5.58, 0.287, 0.135\}$; $\delta_{\text{odd}} = \{2.41, 3.95, 2.53\}$; $\{x_0, v_0\} = \{-0.456, -5.29\}$.

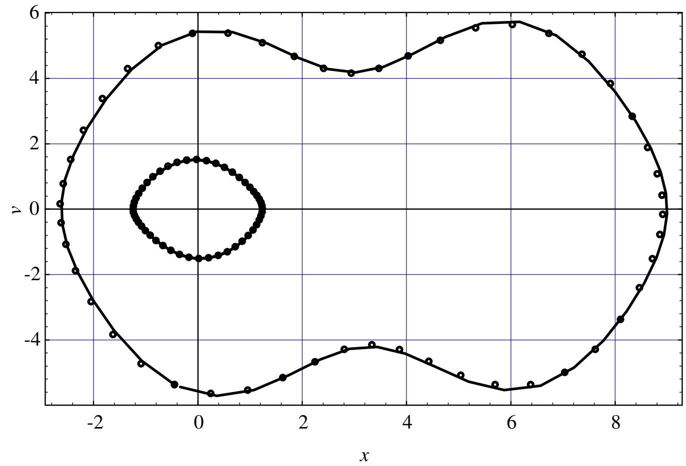

Figure 13. $\epsilon = 0.2$; $a = 0.8$; $f = 0.45$. $C_{\text{odd}} = \{1.3, 0.067, 0.003\}$; $\delta_{\text{odd}} = \{0.15, 0.228, 0.467\}$; $\{x_0, v_0\} = \{-0.212, 1.49\}$. $C_{\text{odd}} = \{5.65, 0.267, 0.139\}$; $\delta_{\text{odd}} = \{2.42, 4., 2.6\}$; $\{|\ell_1|, |\ell_3|, |\ell_5|\} \leqslant \{9, 5, 5\}$; $\{x_0, v_0\} = \{-0.445, -5.37\}$.

By using (3.2) with $\epsilon = 0.2$, $C' \equiv \{C_1, 0, 0, 0, 0, 0\}$ (first-order approximation), we found for different values of f the critical values of external force a, in the sense that, for $a \lesssim a_{\text{critical}}$, there is a stable period 1 solution. There may be more than one; there is at least one. a_{critical} is the value of (3.2) for $C_1 = \pi/2$. The curve is shown in figure 3. This represents very nearly the frontier between the wide, stable period 1 region and the region formed by bands with complex periodic solutions or chaotic solutions shown in figure 2 in the article of Pedersen and Davidson [20].

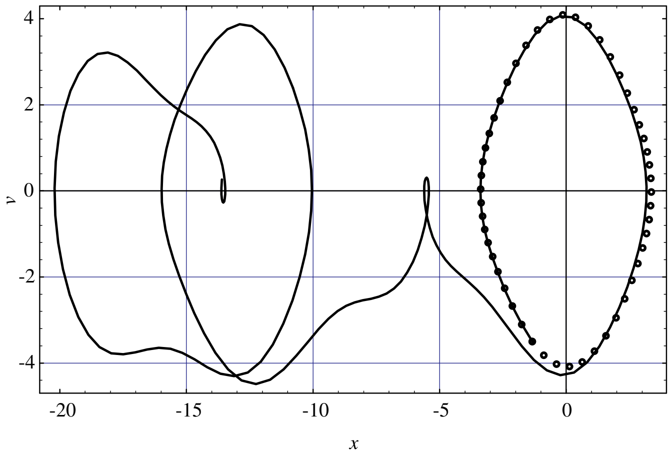

Figure 14. $\epsilon = 0.2$; $a = 1.5$; $f = 0.7$. $C_{\text{odd}} = \{3.51, 0.161, 0.0185\}$; $\delta_{\text{odd}} = \{2.8, 2.03, 1.36\}$; $\{x_0, v_0\} = \{-1.34, -3.5\}$ is a point of the theoretical orbit. This is a rather high-amplitude forcing and the motion is totally unstable.

5. Results of numerical calculations

The present theory is used to find solutions for $\epsilon = 0.2$ and the phase diagrams are compared to results of numerical integration of the original equation (1.2).

In the figures full lines were obtained by numerical integrations and the dots by using this theory.

Sometimes, the basin of the solution is very narrow. The sensitive dependence of the motion on initial conditions makes it difficult to find, numerically, some orbit. We use as initial conditions one point of the analytical solution. In the figures, $\{x_0, v_0\}$ are the initial values of $\{\theta, \dot{\theta}\}$ used for numerical integration.

Even-order coefficients were found to be negligible in our calculations. In the figure captions, only odd-order terms are given: $C_{\text{odd}} \equiv \{C_1, C_3, C_5\}$ and $\delta_{\text{odd}} \equiv \{\delta_1, \delta_3, \delta_5\}$.

According to figure 1, it is possible to have more than one periodic motion for the same values of ϵ, a and f. We found for $a = 0.35$ and $f = 0.7$ three stable solutions, as predicted (figure 4).

For $a = 0.5$ and $f = 0.6$ only one of the solutions ($C_1 \simeq 0.898$) was found to be stable (figure 5).

For $\Theta_0 = 0$, $C_1 \leqslant \pi/2$ the solution must be stable. This falls below the curve in figure 3. We can assert that, in this region, there is, at least, one stable period 1 solution (figures 6–9). Outside this region it is not certain.

For $a = 0.8$ and $f = 0.48$ no stable solution was found.

For $a = 0.52$ and $f = 0.689$ it was possible to reproduce quite exactly the results in the book by Rasband [11] (figure 6.10, p 126). The motion is very sensitive to initial conditions. For slightly different initial conditions numerical integration may converge to a period 3 oscillation. Period 1 and 3 solutions are shown in figure 10.

For $a = 0.52$ and $f = 0.65$ there is a stable solution. For slightly different initial conditions the motion may depart from the orbit; the solution is chaotic (figure 11).

$\Theta_0 = \pi$, represents oscillation around the top position of the pendulum. Not more than one solution was found for each group of parameters (figures 12 and 13). In order to find figure 13, it was necessary to take Bessel functions of much higher order: values of ℓ_1 and ℓ_3 up to nine and five, respectively. These solutions are far more unstable than the others.

According to figure 1, large-amplitude forces (large a) should produce very high-amplitude motions. We could easily find some analytical solutions but numerical integration of (1.2) is very unstable as shown in figure 14. For higher-amplitude motions we have to use higher-order Bessel functions and more terms.

6. Conclusions

A pendulum oscillation contains many harmonics that arise from nonlinearity of the equation of motion. This theory allows us to find systematically all period 1 solutions.

The theory itself is simple but the results are quite remarkable.

- Depending on the viscosity ϵ of the medium, it is possible to delimit the values of system parameters a and f in order to have, at least, one stable period 1 oscillation.
- There is a maximum number of stable solutions for each set of system parameters (ϵ, a, f).
- There may be stable oscillation around the top position of the pendulum.

A similar setup can be extended to find multiple-period solutions ($p = 3$, for example) as well as to analyse other nonlinear equations.

Acknowledgment

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Appendix

Instead of (1.2) we use

$$2\mathrm{i}f^2\ddot{\theta} + 2\mathrm{i}\epsilon f\dot{\theta} + \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = a(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}). \tag{A.1}$$

Each term is written as a complex Fourier series. Thus

$$\begin{aligned} 2\mathrm{i}f^2\ddot{\theta} &= f^2\sum_{m=1}^{\infty} C_m\left(-\frac{m^2}{p^2}\right)(\mathrm{e}^{\mathrm{i}((m/p)t-\delta_m)} - \mathrm{e}^{-\mathrm{i}((m/p)t-\delta_m)}) \qquad \text{and} \\ 2\mathrm{i}\epsilon f\dot{\theta} &= \epsilon f\sum_{m=1}^{\infty} C_m\left(\mathrm{i}\frac{m}{p}\right)(\mathrm{e}^{\mathrm{i}((m/p)t-\delta_m)} + \mathrm{e}^{-\mathrm{i}((m/p)t-\delta_m)}). \end{aligned} \tag{A.2}$$

Equation (2.3) is written

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\theta} &= \mathrm{e}^{\mathrm{i}\Theta_0}\sum_{s=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}st/p}\left[\sum\nolimits^{s} J_{\ell_1}(C_1)J_{\ell_2}(C_2)J_{\ell_3}(C_3)\ldots\mathrm{e}^{-\mathrm{i}\sum\ell_m\delta_m}\right] \\ &= \mathrm{e}^{\mathrm{i}\Theta_0}\sum_{s=1}^{+\infty}\mathrm{e}^{\mathrm{i}st/p}\left[\sum\nolimits^{s} J_{\ell_1}(C_1)J_{\ell_2}(C_2)J_{\ell_3}(C_3)\ldots\mathrm{e}^{-\mathrm{i}\sum\ell_m\delta_m}\right] \\ &\quad +\mathrm{e}^{\mathrm{i}\Theta_0}\sum_{s=1}^{+\infty}\mathrm{e}^{-\mathrm{i}st/p}\left[\sum\nolimits^{s} J_{-\ell_1}(C_1)J_{-\ell_2}(C_2)J_{-\ell_3}(C_3)\ldots\mathrm{e}^{+\mathrm{i}\sum\ell_m\delta_m}\right] \\ &\quad +\mathrm{e}^{\mathrm{i}\Theta_0}\left[\sum\nolimits^{0} J_{\ell_1}(C_1)J_{\ell_2}(C_2)J_{\ell_3}(C_3)\ldots\mathrm{e}^{-\mathrm{i}\sum\ell_m\delta_m}\right]. \end{aligned} \tag{A.3}$$

Using the identity $J_{-\ell} = (-1)^{\ell} J_{\ell}$, we obtain

$$\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\Theta_0} \sum_{s=1}^{+\infty} \mathrm{e}^{\mathrm{i}st/p} \sum\nolimits^{s} J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} + \mathrm{e}^{\mathrm{i}\Theta_0} \sum_{s=1}^{+\infty} \mathrm{e}^{-\mathrm{i}st/p} \sum\nolimits^{s} (-1)^{\ell_t} J^{\ell_t} \mathrm{e}^{+\mathrm{i}\sum \ell_m \delta_m}$$
$$+ \mathrm{e}^{\mathrm{i}\Theta_0} \sum\nolimits^{0} J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} \tag{A.4}$$

where $\ell_t \equiv \sum \ell_m$ and $J^{\ell_t} \equiv J_{\ell_1}(C_1) J_{\ell_2}(C_2) \ldots$, and

$$\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = \sum_{s=1}^{+\infty} \mathrm{e}^{\mathrm{i}st/p} \sum\nolimits^{s} J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} (\mathrm{e}^{\mathrm{i}\Theta_0} - (-1)^{\ell_t} \mathrm{e}^{-\mathrm{i}\Theta_0})$$
$$+ \sum_{s=1}^{+\infty} \mathrm{e}^{-\mathrm{i}st/p} \sum\nolimits^{s} (-1)^{\ell_t} J^{\ell_t} \mathrm{e}^{+\mathrm{i}\sum \ell_m \delta_m} (\mathrm{e}^{\mathrm{i}\Theta_0} - (-1)^{\ell_t} \mathrm{e}^{-\mathrm{i}\Theta_0})$$
$$+ \mathrm{e}^{\mathrm{i}\Theta_0} \sum\nolimits^{0} J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} - \mathrm{e}^{-\mathrm{i}\Theta_0} \sum\nolimits^{0} J^{\ell_t} \mathrm{e}^{+\mathrm{i}\sum \ell_m \delta_m}. \tag{A.5}$$

By substituting (A.2) and (A.5) into (A.1), we obtain one algebraic equation for each harmonic:

- $s > 0;\ s \neq p$

$$\left(-\left(\frac{fs}{p}\right)^2 + \mathrm{i}\epsilon \frac{fs}{p}\right) C_s \mathrm{e}^{-\mathrm{i}\delta_s} + \sum\nolimits^{s} J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} (\mathrm{e}^{\mathrm{i}\Theta_0} - (-1)^{\ell_t} \mathrm{e}^{-\mathrm{i}\Theta_0}) = 0 \tag{A.6}$$

- $s = p$

$$(-f^2 + \mathrm{i}\epsilon f) C_p \mathrm{e}^{-\mathrm{i}\delta_p} + \sum\nolimits^{p} J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} (\mathrm{e}^{\mathrm{i}\Theta_0} - (-1)^{\ell_t} \mathrm{e}^{-\mathrm{i}\Theta_0}) = a \tag{A.7}$$

- $s = 0$

$$\mathrm{e}^{\mathrm{i}\Theta_0} \sum\nolimits^{0} J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} - \mathrm{e}^{-\mathrm{i}\Theta_0} \sum\nolimits^{0} J^{\ell_t} \mathrm{e}^{+\mathrm{i}\sum \ell_m \delta_m} = 0. \tag{A.8}$$

Equations (2.5) and (2.10) are derived at once from (A.6) and (A.7).

In (A.8), $\sum^0$ is the sum over all possible values of $\{\ell_1, \ell_2, \ldots\}$ that satisfy $\ell_1 + 2\ell_2 + 3\ell_3 + \cdots = 0$. If $\{\ell_1, \ell_2, \ldots\}$ is part of the set, then $\{-\ell_1, -\ell_2, \ldots\}$ is also part of it. Therefore

$$\sum\nolimits^{0} J^{\ell_t} \mathrm{e}^{+\mathrm{i}\sum \ell_m \delta_m} \equiv \sum\nolimits^{0} J_{\ell_1}(C_1) J_{\ell_2}(C_2) \ldots \mathrm{e}^{+\mathrm{i}\sum \ell_m \delta_m}$$
$$= \sum\nolimits^{0} J_{-\ell_1}(C_1) J_{-\ell_2}(C_2) \ldots \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m}$$
$$= \sum\nolimits^{0} (-1)^{\ell_t} J_{\ell_1}(C_1) J_{\ell_2}(C_2) \ldots \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} \tag{A.9}$$

and (A.8) will be

$$\sum\nolimits^{0} (\mathrm{e}^{\mathrm{i}\Theta_0} - (-1)^{\ell_t} \mathrm{e}^{-\mathrm{i}\Theta_0}) J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} = 0. \tag{A.10}$$

This is written

$$\sum_{\ell_t = \text{even}}^{0} 2\mathrm{i} \sin \Theta_0 J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} + \sum_{\ell_t = \text{odd}}^{0} 2 \cos \Theta_0 J^{\ell_t} \mathrm{e}^{-\mathrm{i}\sum \ell_m \delta_m} = 0. \tag{A.11}$$

The left-hand side of (A.8) is a real number. The imaginary part is (2.4).

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